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RESEARCH MEMORANDUM

TWO SIMPLE CONTROL POLICIES FOR A MULTICOMPONENT MAINTENANCE SYSTEM

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Abstract

Optimal group maintenance policies for a set of M identical machines subject to stochastic failures are considered. The control of the system is not based on the complete age configuration of all components, nor on the number of failed components only. We compromise between these two extreme cases by introducing for each component four possible states: good, doubtful, preventive maintenance is due and failed. Two types of control policies are considered both based on the number of doubtful components at component failure epochs.
First the model with exponentially distributed sojourn times in the good and doubtful state is considered. Explicit expressions are derived for various performance measures, like the time to system replacement and the average costs per unit time. Next we consider the model in which the sojourn times are governed by a general lifetime distribution for each component. By making use of the results for the exponential model several approximations for the performance measures of the lifetime model are presented. Validation of these approximations is performed by simulation.

Multicomponent maintenance systems are of increasing importance, not only in traditional areas like road maintenance, aircraft industry and oil-production but also in the design and operation of computers and other service facilities. In maintenance optimization models the goal is to find the right compromise between preventive maintenance (which hopefully extends the period of proper operation of the system) and corrective maintenance or replacement (which essentially replaces an old system by a new one). Apart from pure cost considerations also technological developments play an important role in the decision process when to replace a system. This decision process becomes rather involved when the system is composed of many major components that require maintenance. In these situations an important issue is when to combine maintenance or replacement activities on several individual components.

Aspects of this problem have been investigated by several authors in recent literature. VERGIN and SCRIABIN (1977) consider a series system composed of two identical independent components, which are subject to stochastic failures. The optimal maintenance policy (when to maintain a single component and when to maintain both components simultaneously) does not have a simple structure. ÖZEKICI (1988) provides a characterization of the optimal policy for a multi component system with dependent lifetimes. He shows that the optimal policy may have some counter intuitive properties. HAURIE and l'ECUYER (1982) consider a system of M components with identical IFR lifetime distributions. When an individual component fails it is replaced immediately by a new one. At such a replacement opportunity it is possible to replace other (non failed) components simultaneously. The total replacement costs consist of a fixed cost for every time a replacement is carried out and a linear cost in the number of replaced components. It is shown that the optimal policy has a monotonicity property in the following sense. When at a certain age configuration i_k, $1 \le k \le M$ of the components it is decided to replace the whole system then this decision is also optimal in every age configuration j_k with $j_k \ge i_k$, $1 \le k \le M$. However, this monotonicity does not hold with respect to partial replacements of the system, i.e. when for i_k, $1 \le k \le M$ the optimal decision is to replace l components it may occur that the optimal decision in state j_k with $j_k \ge i_k$, $1 \le k \le M$ is to replace less than l components. Due to these phenomena particular attention is paid in

the literature to policies which on one hand have a nice structure (and are easy to implement) and which are on the other hand close to optimality (See e.g. VAN DER DUYN SCHOUTEN and VANNESTE (1990)).
The classes of maintenance policies described above take advantage of the information about the state (age) of every individual system component.
On the other hand several authors studied coordinated group maintenance policies which are based on the number of failed components in the system. (See ASSAF and SHANTIKUMAR (1987) and RITCHKEN and WILSON (1990).)

In this paper we investigate a group replacement policy which recognizes both the advantage and disadvantages of individual component information. On one hand it is obvious that detailed information about the state of each individual component is useful in determining an optimal group replacement policy. On the other hand one has to admit that this detailed information is not always available and, if available, gives rise to optimal policies which are hard to implement.

In this paper we analyse an elementary multi component maintenance model controlled by a simple decision rule. The system that we consider is composed of M identical independently operating components. The condition of each of the components is characterized by four possible states: good (0), doubtful (1), bad (2) and down (3). The sojourn times in each of the individual states is exponentially distributed with parameter depending on the actual state. However, when an individual component enters state 2(3) a preventive (corrective) maintenance is carried out on this single component. The costs of these maintenance operations are given. The maintenance operations are assumed to be instantaneously and the operation of the system is not interrupted. An economic dependency between the components arises by the control rule that is used. In this paper we investigate two different control policies:

Policy A : a complete system replacement is carried out when a single component enters state 2 or 3 and the number of doubtful (state 1) components at that moment is greater than or equal to K;

Policy B : a complete system replacement is carried out at the first time epoch at which an individual component enters state 2 or 3 *after* the first moment at which the number of doubtful components has reached the level K.

The difference between both control rules is rather subtile and concerns the decision to make when the number of doubtful components has reached the level K. Under policy B a system replacement will certainly be performed at the first subsequent epoch at which one of the components turns bad or goes down. However, when this component was already in a doubtful state, a system replacement is not carried out under policy A, since the number of doubtful components decreases from K to K-1.
Policies of type B are in particular of interest when a system replacement needs a lot of organisational preparation. The preparation of the replacement can start as soon as the number of doubtful components has reached the level K, while the replacement is executed at the first subsequent epoch of a component failure (entrance in state 2 or 3).

In this paper we will derive for both type of policies explicit expressions for the average number of system replacements per unit of time as well as the expected number of preventive and corrective component replacements during a system lifetime. With the cost components these expressions provide us with a tool for determining the optimal value of K within both classes of policies.
Although this model can be used to analyse practical maintenance problems, like anti-rust treatment on the piers of a bridge, the maintenance on the different lanes of a highway or controlling the quality of a series system of generators, it most likely oversimplifies many other applications. In many situations a more appropriate description of the stochastic behaviour of components is to attach to every single component a stochastic lifetime with given distribution function. The state of a single component is then described by the (possibly discretized) elapsed lifetime, which usually

gives rise to more then four possible states per individual component. Moreover, the sojourn times in the individual states will no longer be exponentially distributed, but deterministic. We will refer to this model as the "lifetime model" and to the model introduced above as the "exponential model". The lifetime model turns out to be much harder to analyse analytically. Therefore we will provide our exponential model not only with explicit expressions for the quantities of interest but also with approximations, which turn out to be of use for the lifetime model too.

The organisation of the paper is as follows. In section one we describe the model in detail and give some preliminary results concerning the entrance time in an absorbing state of a birth- and death process. In section 2 we provide for a given control policy of type A and B explicit expressions for the average number of system replacements per unit time as well as the expected number of individual component replacements (preventive and corrective) during a system lifetime. In section 3 we derive practically useful approximations for the average costs per unit time under a given control policy of type A or B and we indicate with numerical examples how good these approximations are for the exponential case. Finally we apply in section 4 the approximations to the lifetime model and investigate also for this model the quality of the approximations. The comparisons in the latter case are done by simulation.

1. Model description and preliminaries

We consider a series system, consisting of M independently operating and identical components. The condition of each of the components is characterized by four possible states: good (0), doubtful (1), bad (2) and down (3). Upon entrance in state 2 (3) an immediate preventive (corrective) replacement is carried out, which brings the component back into state 0 without any delay. The sojourn time in state i is exponentially distributed with parameter ν_i, i = 0,1. At the end of a sojourn in state i a transition occurs to either state i+1 or state 3 (down), with probabilities $p_{i,i+1}$ and $p_{i3} = 1-p_{i,i+1}$ respectively.

Note that sojourns in states 2 and 3 are instantaneous because of the immediate preventive and corrective maintenance action. We assume that the complete lifetime of a single component has the following IFR-property:

$$\nu_0 p_{03} < \nu_1 p_{13} < \nu_2$$

i.e. the entrance rate into the down state increases as a function of the present state.

The following cost structure is imposed on the model: for a preventive component replacement a cost c_1 is incurred, whereas a corrective replacement involves cost c_2 ($> c_1$). A system replacement costs c_3. This latter cost may comprise a quantity discount, but also a cost reduction due to technological advancement. We assume

$$c_3 < Mc_1.$$

The objective is to minimize the long-run average cost of the system.

We propose the following maintenance rules, which are referred to as a policies of type A and B respectively:

Policy A: a complete system replacement (or *opportunistic* replacement) is carried out if and only if a single component enters state 2 or 3 and the number of doubtful (state 1) components at that moment is greater than or equal to K;

Policy B: a complete system replacement is carried out at the first time epoch at which an individual component enters state 2 or 3 *after* the first moment at which the number of doubtful components has reached the level K.

Note that for policies of type A detailed information about the state of every single component remains necessary to implement such a policy. When a single component i enters state 2 or 3 at a moment at which the number of doubtful components equals K, it is important to know whether component i came from a good state or a doubtful state. In the first case this situation will give rise to a system replacement, in the latter case not.

For policies of type B it suffices to keep track of the number of doubtful components. Therefore this type of policy is easier to implement and has the advantage that a system replacement is triggered when the number of doubtful components reaches the level K. In this paper we will not investigate under which conditions the optimal maintenance policy is of type A or B.

In the rest of this section we give some preliminary results concerning a continuous time birth- and death process. These results will be used in the analysis in section 2.

Let $\{Y(t), t \geq 0\}$ be a continuous time Markov chain on $\{0,...,L;\Delta\}$ governed by the following transition diagram:

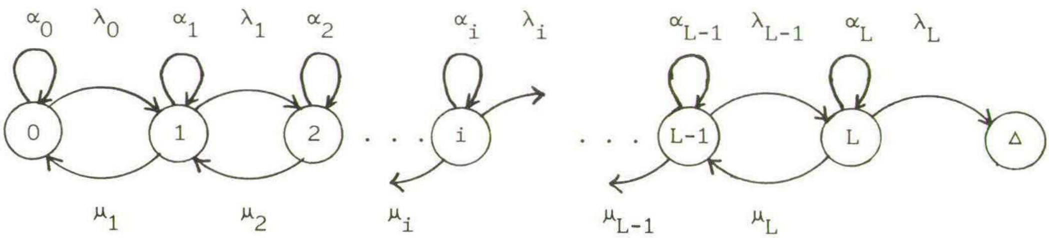

Figure 1. Transition diagram of $\{Y(t), t \geq 0\}$.

The process $\{Y(t), t \geq 0\}$ represents the number of doubtful components under either policy A or B, where state Δ represents the situation in which a system replacement is triggered. The difference between policies A and B is represented by a specification of the values of the transition rates.

We define a "backward jump" of $\{Y(t), t \geq 0\}$ as a transition from some state i to i-1 and a "dummy jump" of $\{Y(t), t \geq 0\}$ as a transition from some state i to itself. Backward jumps correspond to transitions of a single component from "doubtful" via the instanteneous "bad" or "down" state to "good". Dummy jumps correspond to transitions of a single component from "good" via "down" back to "good". Backward jumps are therefore associated with either preventive or corrective replacements, while dummy jumps always correspond to corrective replacements. We are interested in the following quantities:

$\tau_{i,L}$:= expected entrance time of $\{Y(t), t \ge 0\}$ into state Δ, given that $Y(0) = i$, $0 \le i \le L$

$\kappa_{i,L}$:= expected number of backward jumps of $\{Y(t), t \ge 0\}$ before entrance into state Δ, given that $Y(0) = i$, $0 \le i \le L$

$\varphi_{i,L}$:= expected number of dummy jumps of $\{Y(t), t \ge 0\}$ before entrance into state Δ, given that $Y(0) = i$, $0 \le i \le L$.

Explicit expressions for the quantities $\tau_{i,L}$, $\kappa_{i,L}$ and $\varphi_{i,L}$ are obtained in the following theorem (see also KARLIN & TAYLOR (1975), p. 148).

THEOREM 1.1

$$\varphi_{i,L} = \sum_{j=i}^{L} \frac{1}{\lambda_j \rho_j} \sum_{\ell=0}^{j} \alpha_\ell \rho_\ell, \quad 0 \le i \le L \tag{1}$$

$$\kappa_{i,L} = \sum_{j=i}^{L} \frac{1}{\lambda_j \rho_j} \sum_{\ell=0}^{j} \mu_\ell \rho_\ell, \quad 0 \le i \le L \tag{2}$$

$$\tau_{i,L} = \sum_{j=i}^{L} \frac{1}{\lambda_j \rho_j} \sum_{\ell=0}^{j} \rho_\ell \;, \quad 0 \le i \le L \tag{3}$$

where $\rho_0 := \lambda_0^{-1}$, $\rho_1 = \mu_1^{-1}$ and $\rho_i := \dfrac{\lambda_1\lambda_2 \cdots \lambda_{i-1}}{\mu_1\mu_2 \cdots \mu_i}$, $2 \le i \le L$

PROOF. By conditioning on the epoch of the first transition of $\{Y(t), t \ge 0\}$ we get (with $\varphi_{L+1,L} = 0$)

$$\varphi_{i,L} = \frac{\alpha_i}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i}\varphi_{i+1,L} + \frac{\mu_i}{\lambda_i + \mu_i}\varphi_{i-1,L}, \quad 1 \le i \le L \tag{4a}$$

$$\varphi_{0,L} = \alpha_0\lambda_0^{-1} + \varphi_{1,L} \tag{4b}$$

Note that $\alpha_i(\lambda_i + \mu_i)^{-1}$ denotes the expected number of dummy jumps from i to itself before $\{Y(t), t \ge 0)$ jumps from i to either i+1 or i-1.

Define

$$z_i := \varphi_{i,L} - \varphi_{i+1,L}, \quad 0 \le i \le L \tag{5}$$

From (4a) and (4b) we obtain

$$\lambda_i z_i = \alpha_i + \mu_i z_{i-1}, \quad 1 \le i \le L \tag{6a}$$

$$z_0 = \alpha_0 \lambda_0^{-1} \tag{6b}$$

By induction we conclude from (6a) and (6b)

$$z_j = \frac{1}{\lambda_j \rho_j} \sum_{\ell=0}^{j} \alpha_\ell \rho_\ell, \quad 0 \le j \le L \tag{7}$$

Since

$$\varphi_{i,L} = \sum_{j=i}^{L} z_j$$

formula (7) yields (1).

Formulas (2) and (3) follow directly from (1) since $\kappa_{i,L}$ and $\tau_{i,L}$ satisfy relations (4a) and (4b) with α_i replaced by μ_i and 1, respectively, $i \ge 0$. (Here we define $\mu_0 := 0$). □

Finally we consider a second continuous time birth and death process $\{Z(t),\ t \ge 0\}$ on $\{N+1,\dots,M;\delta\}$ with the following transition diagram. $\{Z(t),\ t \ge 0\}$ denotes again the number of doubtful components and δ represents a system replacement. In the sequel N will be chosen to be equal to either K (A-policy) or K-1 (B-policy).

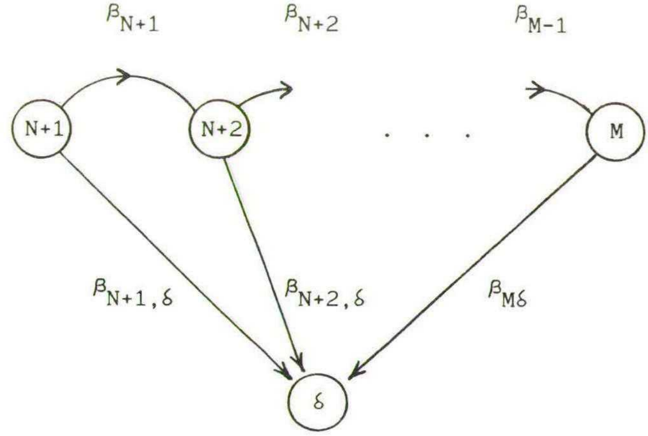

Figure 2. Transition diagram of $\{Z(t),\ t \geq 0\}$.

Let

σ_i := expected entrance time of $\{Z(t),\ t \geq 0\}$ into state δ, given that $Z(0) = i$, $N+1 \leq i \leq M$.

THEOREM 1.2

$$(8) \qquad \sigma_i = \sum_{j=i}^{M} \frac{1}{\beta_j + \beta_{j\delta}} \left[\prod_{\ell=i}^{j-1} \frac{\beta_\ell}{\beta_\ell + \beta_{\ell\delta}} \right], \qquad N+1 \leq i \leq M$$

PROOF. The proof proceeds along the same lines as that of theorem 1.1, starting with the equalities:

$$\sigma_i = \frac{1}{\beta_i + \beta_{i\delta}} + \frac{\beta_i}{\beta_i + \beta_{i\delta}} \sigma_{i+1}, \qquad N+1 \leq i \leq M-1$$

$$\sigma_M = \frac{1}{\beta_{M\delta}}$$

□

(Here we define $\beta_M = 0$.)

2. The average cost analysis of A and B type policies

Let us now consider the series system as described in section 1. As control we choose an A-type policy with parameter K.
Define

$$W(t) := \text{number of doubtful components at time } t,\ t \geq 0.$$

Then $\{W(t),\ t \geq 0\}$ is a continuous time Markov chain on $\{0,\ldots,M\}$. As long as the number of doubtful components has not reached the level K+1 and no system replacement is carried out $\{W(t),\ t \geq 0\}$ behaves like $\{Y(t),\ t \geq 0\}$ with transition diagram as depicted in figure 1 with L = K. Moreover, from the moment on at which the number of doubtful components equals K+1 until system replacement the behaviour of $\{W(t),\ t \geq 0\}$ is similar to that of $\{Z(t),\ t \geq 0\}$ with transition diagram as in figure 2 with N=K.

Referring to figures 1 and 2 we make the following specifications:

$$L = K;\ N = K$$

$$\lambda_i = (M-i)\nu_0 p_{01} \quad ,\quad 0 \leq i \leq K-1$$

$$\lambda_K = (M-K)\nu_0$$

$$\mu_i = i\nu_1 \quad ,\quad 0 \leq i \leq K$$

$$\alpha_i = (M-i)\nu_0 p_{03} \quad ,\quad 0 \leq i \leq K-1$$

$$\alpha_K = 0$$

$$\beta_i = (M-i)\nu_0 p_{01} \quad ,\quad K+1 \leq i \leq M-1$$

$$\beta_{i\delta} = (M-i)\nu_0 p_{03} + i\nu_1, \quad K+1 \leq i \leq M$$

Define

$$X_i(t) := \text{state of component } i \text{ at time } t,\ t \geq 0$$

and

$$X(t) = (X_1(t),\dots,X_M(t))$$

Then $\{X(t),\ t \geq 0\}$ is a regenerative vector-valued stochastic process, with the moments of system replacement as regeneration epochs. Defining a cycle as the time elapsed between two successive system replacements, we conclude from the theory of regenerative processes, that the long run average cost per unit time

$$g_A := \lim_{t\to\infty} \frac{EC_A(t)}{t} = \frac{EC_A(T_A)}{ET_A}, \tag{9}$$

where $C_A(t)$:= the cumulative costs incurred in $[0,t]$
T_A := length of a cycle.

Since we assumed that all components are identical the relevant behaviour of $\{X(t),\ t \geq 0\}$ on $[0,T_A]$ can be completely described by $\{W(t),\ t \geq 0\}$ on $[0,T_A]$, where T_A, by definition, represents the moment of absorption if $\{W(t),\ t \geq 0\}$ into state δ.

THEOREM 2.1

$$g_A = \frac{c_2\varphi_{0,K} + c_2p_{13}\kappa_{0,K} + c_1p_{12}\kappa_{0,K} + c_3}{\tau_{0,K} + p_{01}\sigma_{K+1}} \tag{10}$$

PROOF. Note that T_A can be written as

$$T_A = T_0 + T_1,$$

where

$$T_0 := \text{entrance time of } \{W(t),\ t \geq 0\} \text{ into } \Delta$$

and

$$T_1 := \text{time between entrance of } \{W(t),\ t \geq 0\} \text{ into } \Delta \text{ and entrance into } \delta.$$

i.e., T_0 represents the moment at which $\{W(t),\ t \geq 0\}$ leaves the set $\{0,\ldots,K\}$, while T_1 denotes the time-interval between T_0 and system-replacement.

Note that

$$\mathbb{P}(T_1 = 0) = p_{03}$$

Hence we find

$$ET_A = \tau_{0,K} + p_{01}\sigma_{K+1} \tag{11}$$

On $[0,T_A]$ only costs are incurred on $[0,T_0]$ (costs of corrective and preventive component replacements) and at time T_A (system replacement costs). Every dummy jump of $\{W(t),\ t \geq 0\}$ corresponds to a corrective replacement and every backward jump of $\{W(t),\ t \geq 0\}$ corresponds to a corrective component replacement with probability p_{13} and with a preventive component replacement with probability p_{12}. Hence

$$EC_A(T_A) = c_3 + c_2\varphi_{0,K} + c_2p_{13}\kappa_{0,K} + c_1p_{12}\kappa_{0,K} \tag{12}$$

Combining (9), (11) and (12) yields (10). □

Next we consider our maintenance system controlled by a B-type policy with parameter K. Again we define

$$W(t) := \text{number of doubtful components at time } t,\ t \geq 0.$$

Then $\{W(t),\ t \geq 0\}$ is a continuous time Markov chain on $\{0,\ldots,M\}$. As long as the number of doubtful components has not reached the level K, $\{W(t),\ t \geq 0\}$ behaves like $\{Y(t),\ t \geq 0\}$ with transition diagram as depicted in figure 1 with $L = K-1$. From the moment on at which the number of doubtful components has reached the level K until system replacement $\{W(t),\ t \geq 0\}$ behaves like $\{Z(t),\ t \geq 0\}$ with transition diagram as in figure 2, with the following specifications:

$$L = K-1;\ N = K-1$$

$$\lambda_i = (M-i)\nu_0 p_{01} \quad , \quad 0 \leq i \leq K-1$$

$$\mu_i = i\nu_1 \quad , \quad 0 \leq i \leq K-1$$

$$\alpha_i = (M-i)\nu_0 p_{03} \quad , \quad 0 \leq i \leq K-1$$

$$\beta_i = (M-i)\nu_0 p_{01} \quad , \quad K \leq i \leq M-1$$

$$\beta_{i\delta} = (M-i)\nu_0 p_{03} + i\nu_1, \quad K \leq i \leq M$$

Now the analysis proceeds similarly as in the case of an A-policy. Defining

$$g_B := \lim_{t\to\infty} \frac{EC_B(t)}{t}$$

where

$$C_B(t) := \text{the cumulative costs incurred in } [0,t]$$

we get from the theory of regenerative processes

$$g_B = \frac{EC_B(T_B)}{ET_B} \tag{13}$$

where

T_B := time between two successive system replacements under the B-type policy.

THEOREM 2.2

$$ET_B = \tau_{0,K-1} + \sigma_K \tag{14}$$

and

$$g_B = \frac{c_2\varphi_{0,K-1} + c_2p_{13}\kappa_{0,K-1} + c_1p_{12}\kappa_{0,K-1} + c_3}{\tau_{0,K-1} + \sigma_K} \tag{15}$$

PROOF. Proceeds similarly to that of theorem 2.1. □

We conclude this section with some observations resulting from the analysis of the sections 1 and 2, for the B-policy. First, we establish a relationship between the total number of preventive replacements (TNP) and the total number of corrective replacements (TNC). Secondly, we consider the impact of a single system parameter on the various system measures.

COROLLARY 2.1

$$TNC = \left(\frac{1}{p_{01}p_{12}} - 1\right)TNP + K\left(\frac{1}{p_{01}} - 1\right)$$

PROOF. The proof is based on the following relations:

$$\alpha_i\rho_i = \frac{p_{03}}{p_{01}}\,\mu_{i+1}\rho_{i+1}, \text{ and } \frac{\alpha_i}{\lambda_i} = \frac{p_{03}}{p_{01}}, \qquad i = 0,\ldots,K-1.$$

These relations are easily verified, using the specifications for α_i, λ_i and μ_i. As a consequence, we obtain from (1) and (2):

$$\varphi_{0,K-1} = K\,\frac{p_{03}}{p_{01}} + \frac{p_{03}}{p_{01}}\,\kappa_{0,K-1}.$$

The relations $TNP = p_{12}\kappa_{0,K-1}$ and $TNC = p_{13}\kappa_{0,K-1} + \varphi_{0,K-1}$ conclude the proof. □

An immediate consequence is that TNC > TNP in case $p_{01}p_{12} < \frac{1}{2}$.

COROLLARY 2.2. The following table summarizes the relations between a change in the input parameter and a change of a system measure.

	$\varphi_{0,K-1}$	$\kappa_{0,K-1}$	TNP	TNC	$\tau_{0,K-1}$	σ_K
$M \uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$K \uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$
$p_{01}\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$
$p_{12}\uparrow$	=	=	$\uparrow$	$\downarrow$	=	=
$\nu_0\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\nu_1\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$
$\nu_0\uparrow,\ \nu_1\uparrow,\ \frac{\nu_1}{\nu_0} = c$	=	=	=	=	$\downarrow$	$\downarrow$

PROOF. The method of proof is illustrated by the following example, considering the influence of an increase in p_{01} on $\varphi_{0,K-1}$. Substituting the values for α_i, λ_i and μ_i we obtain from (6):

$$z_i = \left[\frac{1}{p_{01}} - 1\right] + \frac{i}{M-i}.\frac{\nu_1}{\nu_0}.\frac{1}{p_{01}}.z_{i-1} \text{ and } z_0 = \frac{1}{p_{01}} - 1.$$

Therefore, as $p_{01}\uparrow$, $z_i\downarrow$, $i = 0,\dots,K-1$ and $\varphi_{0,K-1} = \sum_{i=0}^{K-1} z_i\downarrow$. The impacts on the quantities $\kappa_{0,K-1}$ and $\tau_{0,K-1}$ are obtained in the same manner, whereas the results for TNP and TNC follow from $\varphi_{0,K-1}$ and $\kappa_{0,K-1}$. (See the proof of Corollary 2.1.)

To obtain the results for σ_K we use another method of proof. Notice that

$$\sigma_K = \int_0^\infty P(T_1 > t)dt;\ P(T_1 > t) = \bar{F}(t)^K\, \bar{G}(t)^{M-K} = \left[\frac{\bar{F}(t)}{\bar{G}(t)}\right]^K \bar{G}(t)^M,$$

with

$$\bar{F}(t) = e^{-\nu_1 t} \text{ and } \bar{G}(t) = e^{-\nu_0 t} + \int_{s=0}^{t} p_{01} e^{-\nu_1(t-s)} \nu_0 e^{-\nu_0 s} ds.$$

From this expression, the results with respect to ν_1, p_{01} and p_{12} follow immediately. Evaluating the integral, we obtain,

$$\bar{G}(t) = e^{-\nu_1 t} - p(e^{-\nu_0 t} - e^{-\nu_1 t}), \text{ with } p = \frac{\nu_1 - \nu_0 p_{03}}{\nu_1 - \nu_0}.$$

This implies $1 > \bar{G}(t) \geq e^{-\nu_1 t} = \bar{F}(t)$, which yields the results with respect to K and M. The influence of a change in ν_0 follows from

$$\bar{G}(t) = e^{-\nu_0 t} + p_{01} \int_{u=0}^{t} e^{-\nu_0(t-u)} \nu_1 e^{-\nu_1 u} du + p_{01} e^{-\nu_1 t}. \qquad \square$$

3. Approximations

In section 2 we obtained explicit expressions for the average costs per unit time under a given policy of either type A or B. From these expressions optimal control limits can be obtained. To relax our assumptions about the exponential distributed sojourn times of individual components in their various states, we propose in this section an approximation for the average costs per unit time under a given policy. This approximation will be used in the next section in the analysis of the model in which the sojourn times in the various states are generated by a lifetime distribution of an individual component.

The following analysis is valid irrespectively the type of policy used. We will focus on the system governed by a type B policy. The analysis for a type A policy proceeds similarly. Assume that a system replacement is carried out at time epoch 0. As before we define

$W(t)$:= number of doubtful components at t, $t \geq 0$

Let

$N_p^{(i)}(t)$:= cumulative number of preventive replacements of component i in $[0,t]$; $t \geq 0$, $1 \leq i \leq M$

$N_c^{(i)}(t)$:= cumulative number of corrective replacements of component i in $[0,t]$; $t \geq 0$, $1 \leq i \leq M$

T_B := epoch of first system replacement after 0

and

T_0 := entrance time of $\{W(t),\ t \geq 0\}$ into Δ

From the regenerative analysis given in section 2 it follows that

$$g_B = \frac{c_1 \sum_{i=1}^{M} EN_p^{(i)}(T_B) + c_2 \sum_{i=1}^{M} EN_c^{(i)}(T_B) + c_3}{ET_B} \tag{16}$$

Since the individual components are identical and behave independently on the interval $[0,T_B]$ (16) implies

$$g_B = \frac{Mc_1 EN_p^{(1)}(T_B) + Mc_2 EN_c^{(1)}(T_B) + c_3}{ET_B} \tag{17}$$

In the sequel $N_p^{(1)}(t)$ and $N_c^{(1)}(t)$ will be denoted by $N_p(t)$ and $N_c(t)$ respectively. From the definition of T_0 it follows that between T_0 and T_B no preventive or corrective component replacements are carried out. Hence

$$g_B = \frac{Mc_1 EN_p(T_0) + Mc_2 EN_c(T_0) + c_3}{ET_B} \tag{18}$$

Based on (18) we propose the following approximation $g_B^{(a)}$ for g_B:

$$g_B^{(a)} := \frac{Mc_1 M_p(ET_0) + Mc_2 M_c(ET_0) + c_3}{ET_B}, \tag{19}$$

where $M_p(t)$ and $M_c(t)$ denote the renewal functions associated with the renewal processes $\{N_p(t),\ t \geq 0\}$ and $\{N_c(t),\ t \geq 0\}$, respectively.

Motivation for the approximation (19) is provided by the following arguments. Note that T_0 (the entrance time of $\{W(t), t \geq 0\}$ into Δ) depends on all renewal processes $\{N_p^{(i)}(t), t \geq 0\}$ and $\{N_c^{(i)}(t), t \geq 0\}$, $1 \leq i \leq M$. On the other hand it is intuitively clear that the dependency of T_0 on each individual renewal process will be relatively weak when M and K are not too small. When T_0 is independent of $\{N_p(t), t \geq 0\}$ then g_B can be obtained from

$$EN_p(T_0) = \int_0^{\infty} M_p(t)dG_{T_0}(t) \tag{20}$$

However, computation of the right hand side of (20) has two disadvantages. In the first place complete knowledge of $M_p(t)$ over the range $[0,\infty)$ is necessary and secondly complete knowledge of the distribution function $G_{T_0}(t)$ of T_0 is required.
On the other hand, use of the approximation (19) only requires ET_0 and the computation of the renewal function in one single point.
In the approximation for the lifetime model that we will deal with in the next section both advantages apply. In the approximation of the exponential model the renewal functions $M_p(t)$ and $M_c(t)$ can be obtained explicitly as we will show below. So in this case only the first advantage holds.
Some further theoretical motivation for (19) as approximation for (18) is provided in appendix A.

The time between two preventive replacements of an individual component on $[0,T_0]$ can be considered as the entrance time into the absorbing state 2 of a continuous time Markov chain on $\{0,1,2\}$ governed by the transition diagram as depicted in figure 3.

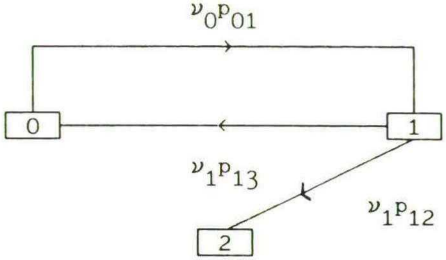

Figure 3. Transition diagram of single component until preventive replacement.

In figure 3 state 0 denotes the component in good condition, state 1 the component in doubtful condition. Note that transitions from state 1 to state 0 occur due to corrective replacements.
From this representation it follows that the time between two successive preventive replacements of an individual component has a phase-type distribution F_p (cf. NEUTS (1981)) of the following form

$$1 - F_p(t) = p_1 e^{-\mu_1 t} + p_2 e^{-\mu_2 t}$$

with $\mu_1 > 0$, $\mu_2 > 0$, $p_1 > 0$, $p_2 < 0$ and $p_1 + p_2 = 1$.

This distribution belongs to the class of K_2-distributions (cf. TIJMS (1986), p. 400).
The details of this derivation are given in appendix B.

The renewal function $M_p(t)$ generated by $F_p(t)$ is given by (see TIJMS (1986), pp. 74 and 399-400)

$$M_p(t) = \frac{t}{\mu_p} + \frac{1}{2}(c_v^2 - 1)(1 - \exp\{-(p_1\mu_2 + p_2\mu_1)t\}), \quad t \geq 0$$

where

$$\mu_p := \frac{p_1}{\mu_1} + \frac{p_2}{\mu_2}.$$

For the values of p_1, p_2, μ_1 and μ_2 we refer to appendix B.

The derivation of $F_c(t)$ and $M_c(t)$, the distribution of the time between two successive corrective replacements and its corresponding renewal function, proceeds along the same lines, starting with the transition diagram:

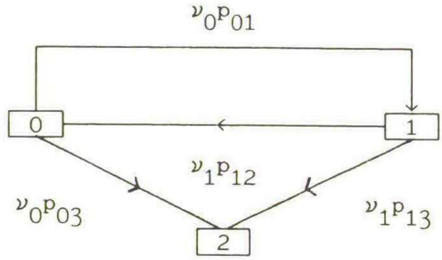

Figure 4. Transition diagram of single component until corrective replacement.

Note that now transitions from doubtful to good are generated by preventive replacements. For details we again refer to appendix B.

We conclude this section with a validation of this approximation. In table I we present for a certain choice of the system parameters (c_1, c_2, c_3; M, K; ν_0, ν_1, p_{03}, p_{13}) the following quantities

$$EN_p(T_0),\ M_p(ET_0);\ EN_c(T_0),\ M_c(ET_0);\ g_B,\ g_B^{(a)},\ d := \frac{g_B^{(a)} - g_B}{g_B};\ ET_0 \text{ and } ET.$$

In table II we present some statistical results about the numerical experiments we performed. For 330 different choices of system parameters the table shows the number of times that d took a value in different intervals. Apart from variation in M and K as shown in table I we considered different values for ν_1 (0,5; 1; 1,5; 2), p_{01} (0.95; 0.85; 0.70) and p_{12} (0.85; 0.70; 0.40) while maintaining the IFR-property $\nu_0 p_{03} < \nu_1 p_{13}$.

	K	$EN_p(T_0)$	$M_p(ET_0)$	$EN_c(T_0)$	$M_c(ET_0)$	g_B	$g_B^{(a)}$	d	ET_0	ET
M = 4	1	0.00	0.03	0.05	0.05	3.51	3.81	0.09	0.29	0.67
	2	0.10	0.19	0.16	0.19	3.26	3.72	0.14	0.86	1.13
	3	0.59	0.77	0.54	0.59	3.27	3.68	0.13	2.44	2.65
	4	4.12	4.40	2.98	3.06	3.47	3.62	0.04	12.02	12.18
M = 8	2	0.02	0.04	0.06	0.07	9.17	9.63	0.05	0.35	0.56
	4	0.25	0.32	0.26	0.28	7.30	7.95	0.09	1.24	1.38
	6	2.23	2.41	1.65	1.70	7.03	7.35	0.05	6.67	6.87
	7	10.88	11.11	7.56	7.63	7.12	7.22	0.01	29.69	29.78
M = 16	1	0.00	0.00	0.01	0.01	33.60	33.83	0.01	0.07	0.25
	2	0.01	0.01	0.03	0.03	28.83	29.24	0.01	0.16	0.31
	3	0.02	0.03	0.05	0.05	24.66	25.20	0.02	0.27	0.39
	4	0.04	0.05	0.07	0.08	21.37	22.02	0.03	0.40	0.51
	5	0.08	0.10	0.11	0.12	18.90	19.63	0.04	0.58	0.68
	6	0.15	0.18	0.17	0.17	17.11	17.91	0.05	0.83	0.91
	7	0.26	0.31	0.26	0.27	15.86	16.70	0.05	1.21	1.29
	8	0.48	0.55	0.42	0.44	15.04	15.87	0.06	1.86	1.93
	9	0.95	1.04	0.75	0.77	14.57	15.27	0.05	3.16	3.23
	10	2.11	2.22	1.54	1.58	14.36	14.83	0.03	6.28	6.34
	11	5.64	5.78	3.96	4.00	14.31	14.55	0.02	15.65	15.70
	12	20.07	19.68	13.41	13.46	14.33	14.41	0.01	52.25	52.31
	13	93.20	93.40	63.58	63.64	14.34	14.36	0.00	246.35	246.40
	14	657.02	657.25	447.37	447.43	14.35	14.35	0.00	1730.99	1731.04
	15	7622.41	7622.67	5188.53	5188.60	14.35	14.35	0.00	20071	20071
	16	192002	192002	130690	130690	14.35	14.35	0.00	505552	505552
M = 32	8	0.05	0.06	0.08	0.08	46.23	47.04	0.02	0.44	0.50
	16	0.99	1.06	0.77	0.78	29.84	30.80	0.03	3.21	3.24
	24	1890.40	1890.57	1286.87	1286.92	28.70	28.70	0.00	4978.40	4978.43

Table I. Exact and approximate values of average costs
$(c_1, c_2, c_3; \nu_0, \nu_1, p_{01}, p_{12}) = (1, 2, 0.5M; 1, 1.5, 0.85, 0.70)$.

	d					
M	0-5%	5-10%	10-15%	15-20%	20-25%	total
4	21	29	21	13	4	88
8	54	27	7	-	-	88
16	77	11	-	-	-	88
32	66	-	-	-	-	66
						330

Table II. Values of d for 330 different models.

Our general impression is that the deviation d decreases with increasing value of M, while the deviation is maximal at K = M/2. Although in all our numerical experiments the deviation d turned out to be positive, which makes $g_B^{(a)}$ an upperbound for g_B, it is not quite obvious that this conclusion can be generally drawn. In fact proposition A3 and the analysis in appendix B imply, in case the assumption of independence between T_0 and $\{N_p(t), t \geq 0\}$ and $\{N_c(t), t \geq 0\}$ would hold, that $g_B^{(a)} \leq g_B$.

4. Approximations for the lifetime model

In this section we consider the situation in which the aging of individual components is described by a general lifetime distribution G. This model can be brought on equal footing with the model described in section 1 by introducing two critical age parameters r and R (R ≥ r), with the following interpretation. A component with age less than r is considered as being good (state 0); when the age is between the values r and R the component is doubtful (state 1). When the age of a component reaches the value R a preventive maintenance is carried out (instantaneous state 2). Finally, if the component fails before age R it is replaced correctively (instantaneous state 3).

From the lifetime distribution G and the values r and R the relevant transition probabilities between various states p_{01} and p_{12} are obtained as follows.

$$p_{01} = 1 - G(r) \tag{21}$$

and

$$p_{12} = \frac{1 - G(R)}{1 - G(r)} \tag{22}$$

The sojourn times L_0 and L_1 of an individual component in the states 0 and 1 are distributed according to

$$\mathbb{P}(L_0 = r) = 1 - G(r)$$

$$\mathbb{P}(L_0 > s) = 1 - G(s), \quad 0 \le s < r$$

$$\mathbb{P}(L_1 = R-r) = \frac{1 - G(R)}{1 - G(r)}$$

$$\mathbb{P}(L_1 > s) = \frac{1 - G(s+r)}{1 - G(r)}, \quad 0 \le s < R-r$$

Hence the expected sojourn times are given by

$$EL_0 = \int_0^r \mathbb{P}(L_0 > s)ds = \int_0^r (1 - G(s))ds$$

and

$$EL_1 = \int_0^{R-r} \mathbb{P}(L_1 > s)ds = \int_0^{R-r} \frac{1 - G(s+r)}{1 - G(r)}ds = \frac{1}{1 - G(r)} \int_r^R (1 - G(s))ds.$$

In comparing the aging model with the exponential model we therefore choose the following transition rates:

$$\nu_0^{-1} = \int_0^r (1 - G(s))ds \tag{23}$$

and

$$\nu_1^{-1} = \frac{1}{1 - G(r)} \int_r^R (1 - G(s))ds. \tag{24}$$

The difference between the lifetime model and the exponential model described in section 1 is that the sojourn times in states 0 and 1 are no longer exponential distributed. Moreover the transition mechanism between the various states is no longer independent of the sojourn times, e.g. when component i enters at time 0 state 0, we have

$$P(X_i(r) = 1 | L_0 = r) = 1$$

where

$$X_i(t) := \text{state of component } i \text{ at time } t,\ t \geq 0.$$

In order to get approximations for the average costs per unit time under either a type A or B policy for the lifetime model, we investigate the use of formulas (10), (11) and (14), (15) for g_A, ET_A and g_B, ET_B respectively. We will restrict our attention to policies of type B.

This basic approximation turns out to yield very poor results. Deviations of 30% for g_B are not exceptional. (The comparison was made by simulation as will be done throughout this section.)

As a first improvement of this basic approximation we investigate the use of formula (19) in which the renewal functions $M_p(.)$ and $M_c(.)$ are based on the lifetime distribution G itself.
The interrenewal distributions between consecutive preventive and consecutive corrective replacements, $F_p(t)$ and $F_c(t)$ respectively are derived as follows.
Let X, X_p and X_c be generic random variables denoting the time between two consecutive replacements (X), two consecutive preventive replacements (X_p) and two consecutive corrective replacements (X_c) respectively, for one single component, before system replacement. The corresponding renewal functions are $M(t)$, $M_p(t)$ and $M_c(t)$

We have

$$P(X > t) = \begin{cases} 1 - G(t), & 0 \le t \le R \\ 0 & , \quad t > R \end{cases}$$

with

$$EX = \int_0^R (1 - G(t))dt$$

$$EX^2 = 2\int_0^R t(1 - G(t))dt$$

and

$$P(X_c > kR + t) = (1 - G(R)^k(1 - G(t)), \quad 0 \le t < R$$

with

$$EX_c = \frac{1}{G(R)} \int_0^R (1 - G(t))dt$$

$$EX_c^2 = \frac{2}{G(R)} \int_0^R t(1 - G(t))dt + \frac{2R(1 - G(R))}{(G(R))^2} \int_0^R (1 - G(t))dt.$$

The computation of $M(ET_0)$ and $M_c(ET_0)$ is now carried out by the procedure proposed by ROSS (1987). $M_p(ET_0)$ is obtained from the equality

$$M(t) = M_p(t) + M_c(t), \quad t \ge 0.$$

The approximation of ET_0 and ET_B is still based on the exact analysis of the exponential model, i.e.

$$ET_0^{(1)} := \tau_{0,K-1}$$

and

$$ET_B^{(1)} := \tau_{0,K-1} + \sigma_K$$

where $\tau_{0,K-1}$ and σ_K are obtained from (3) and (8) respectively, while the parameters ν_0, ν_1, p_{01} and p_{12} are determined by (21) upto (24).

<u>REMARK</u>. If $ET_0^{(1)}$ is fairly large compared to EX_c and EX_p a good alternative for Ross' procedure is the use of the asymptotic expansions of M(t), $M_p(t)$ and $M_c(t)$:

$$M(t) \approx \frac{t}{EX} + \frac{EX^2}{2(EX)^2} - 1$$

$$M_C(t) \approx \frac{t}{EX_c} + \frac{EX_c^2}{2(EX_c)^2} - 1$$

$$M_p(t) = M(t) - M_c(t)$$

We refer to TIJMS (1986), pp. 7 for rules of thumb under which these approximations apply. In our numerical examples we have applied the asymptotic expansions when $ET_0^{(1)} \geq 3EX_c$ or $ET_0^{(1)} \geq 3EX_p$ for corrective and preventive replacement respectively.

This first improvement turned out to yield better results for g_B, although the approximation for ET_0 and ET_B still have a poor performance (See first and second column in table III).

ET_0 sim		$ET_0^{(1)}$	$ET_0^{(2)}$	$ET_0^{(3)}$
0.17	(0.00)	0.50	0.17	0.17
0.26	(0.01)	2.82	0.34	0.31
8.98	(1.52)	83.44	12.78	9.95
0.26	(0.00)	0.86	0.27	0.27
1.15	(0.12)	8.39	2.16	1.95
219.00	(14.30)	461.63	258.10	238.94
0.37	(0.00)	1.44	0.45	0.44
8.64	(0.58)	21.15	11.05	10.57
2.235E3	(94.6)	2.789E3	2.429E3	2.381E3
0.75	(0.03)	2.73	1.05	1.03
51.82	(2.30)	77.27	60.86	59.94
1.85	(0.09)	4.96	2.56	2.52
236.85	(9.15)	274.49	253.59	252.33

Table III. Comparison of improved approximations of ET_0 with simulated values for Weibull distributed lifetimes.

<u>REMARK</u>. It should be realized that the influence of deviations from the exact value of ET_0 on g_B decreases with increasing ET_0. Referring to (19) we note that for ET_0 large we have $\frac{ET_0}{ET_B} \approx 1$, $\frac{c_3}{ET_B} \approx 0$, $\frac{M_p(ET_0)}{ET_B} \approx \frac{1}{EX_p}$ and $\frac{M_c(ET_0)}{ET_B} \approx \frac{1}{EX_c}$.

Next we present an improvement of the approximations $ET_0^{(1)}$ and $ET_B^{(1)}$. We note that for systems with high reliable components there is a fairly high probability that, starting with a new system at time 0, there are K or more components entering the doubtful state simultaneously at time epoch r. Under these circumstances we therefore have $T_0 = r$ (Recall that T_0 denotes the first epoch after 0 at which the number of doubtful components exceeds K-1).

Suppose, as usual, that at time 0, a system replacement occurs. Define

$$d(i) := \mathbb{P}(W(r) = i).$$

Then it follows that

$$d(i) = \binom{M}{i}(1 - G(r))^{i}G(r)^{M-i}$$

We propose the following approximation for ET_0

$$ET_0^{(2)} := r + \sum_{i=0}^{K-1} d(i)\tau_{i,K-1} \tag{25}$$

This approximation is based on the observation that on the event $\{W(r) = i\}$ there are i components entering the doubtful state 1 at epoch r, while the other M-i components are still in state 0 (due to corrective replacements during $[0,r)$). So the time between r and T_0 can be considered as the entrance time of $\{W(t), t \geq 0\}$ into state K starting in state i at time 0, which is approximated by the corresponding quantity $\tau_{i,K-1}$ of the exponential model.

Next we consider the approximation for ET_B. Define

$$R_B := T_B - T_0 \tag{26}$$

then

$$ER_B = E(E(R_B|W(r)) \tag{27}$$

$$= \sum_{i=0}^{K-1} d(i)E(R_B|W(r) = i) + \sum_{i=K}^{M} d(i)E(R_B|W(r) = i).$$

For the first term on the right hand side of (27) we propose the following approximation:

$$\sum_{i=0}^{K-1} d(i)E(R_B|W(r) = i) \approx \sigma_K \sum_{i=0}^{K-1} d(i) \tag{28}$$

With respect to the second term

$$\sum_{i=K}^{M} d(i)\; E(R_B \mid W(r) = i) = \tag{29}$$

$$= \sum_{i=K}^{M} d(i) \int_0^{R-r} \mathbb{P}(R_B > t \mid W(r) = i)dt$$

the following observation holds. If $G \in$ IFR then

$$Z_\ell := \int_0^{R-r} \left[\frac{1 - G(t+r)}{1 - G(r)}\right]^M dt \sum_{i=K}^{M} d(i) \le \tag{30}$$

$$\le \sum_{i=K}^{M} d(i)E(R \mid W(r) = i) \le$$

$$\le \sum_{i=K}^{M} d(i) \int_0^{R-r} \left[\frac{1 - G(t+r)}{1 - G(r)}\right]^i (1-G(t))^{M-i} dt =: Z_u$$

Note that Z_u corresponds with the situation that the (M-i) components that are not doubtful at epoch r are just entering state 0 at epoch r, while Z_ℓ represents the situation in which those (M-i) components are about leaving state 0 at epoch r. We propose as approximation for ET_B:

$$ET_B^{(2)} := ET_0^{(2)} + \sigma_K \sum_{i=0}^{K-1} d(i) + \frac{1}{2}(Z_u + Z_\ell) \tag{31}$$

and for g_B:

$$g_B^{(2)} := \frac{Mc_1M_p(ET_0^{(2)}) + Mc_2M_c(ET_0^{(2)}) + c_3}{ET_B^{(2)}} \tag{32}$$

The performance of this approximation is fairly good. In table III we present the approximated values $ET_0^{(2)}$ and $ET_B^{(2)}$ together with simulation results for the case of a Weibull distributed lifetime.

Finally we present a further refinement of the approximation of ET_0, which is based on a separate treatment of the event $\{W(r) = K - 1\}$.

$$ET_0^{(3)} := r + \sum_{i=0}^{K-2} d(i)\tau_{i,K-1} + d(K-1)E^{(a)}(D) \tag{33}$$

In (33) $E^{(a)}(D)$ represents an approximation of the expectation of D, the time between r and the first epoch at which $W(t) = K$ given that $K-1$ components just entered the doubtful state at epoch r.

To obtain an explicit expression for $E^{(a)}(D)$ we introduce the random variables:

L_1 := time between r and the first epoch at which one of the $K-1$ components, which became doubtful at r, fails or reaches age R

L_2 := time between r and the first epoch at which one of the $(M-K+1)$ components, which are in state 0 at r, becomes doubtful.

As an approximation for ED we propose

$$E^{(a)}(D) = E\min(L_1,L_2) + P(L_1 \le L_2)\tau_{K-2,K-1} \tag{34}$$

On the event $\{L_1 \le L_2\}$ we have

$D = L_1$ + "entrance time of $\{W(t),\ t \ge 0\}$ into K, starting at time 0 with $K-2$ components in doubtful condition (each with age $r + L_1$)"

We use $\tau_{K-2,K-1}$ as approximation for this entrance time.

On the other hand we note that $D = L_2$ on the event $\{L_1 > L_2\}$.

What remains is to obtain explicit expressions for $E\min(L_1,L_2)$ and $P(L_1 \le L_2)$. We note that

$$P(L_1 > t) = \left[\frac{1 - G(r+t)}{1 - G(r)}\right]^{K-1}, \quad 0 \le t \le R-r \tag{35}$$

To obtain the distribution of L_2 we disregard the possibility that the first failing component, fails again before becoming doubtful. With this simplification, each of the (M-K+1) components in state 0 at epoch r will reach the doubtful state after t+r, if and only if the first failing component on [0,r] fails after t. Therefore, we get

$$\mathbb{P}(L_2 > t) = \left[1 - \frac{G(t)}{G(r)}\right]^{M-K+1} \quad , \quad 0 \le t \le r. \tag{36}$$

From (35) and (36) we obtain

$$E \min(L_1, L_2) = \tag{37}$$

$$= \int_0^{\min(r,R-r)} \mathbb{P}(L_1 > t)\mathbb{P}(L_2 > t)dt$$

$$= \int_0^{\min(r,R-r)} \left[\frac{1 - G(r+t)}{1 - G(r)}\right]^{K-1} \left[\frac{G(r) - G(t)}{G(r)}\right]^{M-K+1} dt$$

and

$$\mathbb{P}(L_1 \le L_2) = \int_0^r \frac{M-K+1}{G(r)} \left[\frac{G(r) - G(t)}{G(r)}\right]^{M-K} g(t) \left[1 - \left[\frac{1 - G(r+t)}{1 - G(r)}\right]^{K-1}\right] dt \tag{38}$$

where g(.) denotes the derivative of G(.).
Together (33), (34), (37) and (38) yield a refinement $ET_0^{(3)}$ of the approximation for ET_0. In table III the performances of $ET_0^{(1)}$, $ET_0^{(2)}$ and $ET_0^{(3)}$ are compared with simulation results.

Finally we present some numerical examples concerning the performance of the best approximations $g_B^{(3)}$ for g_B which is based on the use of $ET_0^{(3)}$ and $ET_B^{(3)}$, which is in accordance with (31) defined by

$$ET_B^{(3)} := ET_0^{(3)} + \sigma_K \sum_{i=0}^{K-1} d(i) + \frac{1}{2} (Z_u + Z_\ell) \tag{39}$$

while

$$g_B^{(3)} := \frac{Mc_1 M_p(ET_0^{(3)}) + Mc_2 M_c(ET_0^{(3)}) + c_3}{ET_B^{(3)}}. \tag{40}$$

We made the following specifications concerning the system. The lifetime is represented by the Weibull distribution with scale parameter λ and shape parameter α. For the system parameters $(M, K; \lambda, \alpha)$ we chose the values, which are exhibited in table IV, together with the mean μ and coefficient of variation c_v^2 for the corresponding Weibull distribution.

no	M	K	λ	α	μ	c_v^2
1-15	16	12	1	2	0.89	0.28
16	16	12	1	1.4	0.91	0.53
17	16	12	1	3	0.89	0.13
18	16	14	1	2	0.89	0.28
19	8	4	1	1	0.89	0.28

Table IV. Parameter values and characteristics of the Weibull distribution for the models in table V.

The values for r and R are chosen as follows: R = 0.5, 0.75, 1.0, 1.25, 1.5 and $r = \frac{1}{3}R, \frac{1}{2}R, \frac{2}{3}R$. As before, the costs parameters were kept fixed at $(c_1, c_2, c_3) = (1, 2, \frac{1}{2}M)$. Together, we obtained 75 configurations of system parameters for the numerical experiments. Detailed results for some of these are given in table V, whereas some global results are reported in table VI.

In table V we give values for the input parameters (r,R) (the other system parameters are exhibited in table IV and they are referred to by the model number), the transition probabilities p_{01} and p_{12}, the probability that $T_0 = r$, the quantity $\tau_{0,K-1}$ from the exponential model, and both the simulated as well as the approximated values for the quantities ET_0, ER_B, the

no.	r	p_{01}	$P(T_0=r)$	ET_0 sim	ER_B sim	TNP sim	TNC sim	g sim
	R	p_{12}	$\tau_{0,K-1}$	$ET_0^{(3)}$	$ER_B^{(3)}$	$M.M_p(ET_0^{(3)})$	$M.M_c(ET_0^{(3)})$	$g^{(3)}$
1.	0.17	0.97	1.00	(0.17; 0.17)	0.11	0.00	0.47	(31.04; 31.80)
	0.50	0.80	0.45	0.17	0.12	0.00	0.48	31.14
2.	0.25	0.94	1.00	(0.25; 0.25)	0.10	0.00	0.99	(28.54; 29.12)
	0.50	0.83	2.19	0.25	0.09	0.00	1.03	29.13
3.	0.33	0.90	0.98	(0.89; 1.73)	0.07	26.3	9.22	(36.89; 39.49)
	0.50	0.87	53.4	0.97	0.07	21.5	4.34	36.50
4.	0.25	0.94	1.00	(0.25; 0.25)	0.10	0.00	0.94	(28.10; 28.70)
	0.75	0.61	0.75	0.25	0.10	0.00	1.02	28.92
5.	0.38	0.87	0.95	(0.45; 0.54)	0.08	1.50	3.44	(28.12; 29.32)
	0.75	0.66	6.21	0.59	0.07	0.82	4.95	28.47
6.	0.50	0.78	0.73	(65.5; 80.5)	0.05	1.05E3	7.95E2	(36.23; 36.23)
	0.75	0.73	3.07E2	71.9	0.05	8.88E2	6.74E2	36.27
7.	0.33	0.90	0.98	(0.33; 0.33)	0.09	0.00	1.70	(27.08; 27.60)
	1.0	0.41	1.25	0.34	0.08	0.00	1.84	27.76
8.	0.5	0.78	0.73	(3.05; 3.65)	0.05	21.6	42.3	(33.37; 33.71)
	1.0	0.47	17.6	4.23	0.05	28.8	54.9	34.23
9	0.67	0.64	0.26	(1.89E3; 2.07E3)	0.03	1.56E5	2.68E5	(34.96; 34.96)
	1.0	0.58	2.77E3	1.98E3	0.03	1.56E5	2.68E5	34.97
10.	0.42	0.84	0.90	(0.51; 0.51)	0.07	0.18	3.97	(27.61; 28.25)
	1.25	0.25	2.54	0.57	0.07	0.00	4.72	27.62

no.	r	p_{01}	$P(T_0=r)$	ET_0sim	ER_Bsim	TNP sim	TNC sim	g sim
	R	p_{12}	$\tau_{0,K-1}$	$ET_0^{(3)}$	$ER_B^{(3)}$	$M.M_p(ET_0^{(3)})$	$M.M_c(ET_0^{(3)})$	$g^{(3)}$
11.	0.63	0.67	0.36	(48.8; 54.2)	0.04	2.07E2	7.91E2	(34.89; 34.93)
	1.25	0.31	93.8	55.6	0.04	2.25E2	8.56E2	34.97
12.	0.83	0.50	0.04	(*)				
	1.25	0.42	3.38E4	3.23E4	0.02	1.32E5	4.99E5	35.02
13.	0.50	0.78	0.73	(1.18; 1.38)	0.06	1.13	16.4	(31.04; 31.68)
	1.5	0.14	5.45	1.51	0.06	0.90	21.0	32.58
14.	0.75	0.57	0.11	(5.22E2; 5.65E2)	0.03	1.07E3	9.08E3	(35.39; 35.39)
	1.5	0.18	6.58E2	5.68E2	0.03	1.12E3	9.50E3	35.40
15.	1.0	0.37	0.00	(*)				
	1.5	0.29	1.34E6	1.33E6	0.02	2.63E6	2.23E7	35.41
16.	0.5	0.68	0.40	(8.06; 9.22)	0.04	67.8	1.25E2	(37.50; 37.66)
	1.0	0.54	21.2	10.57	0.04	85.4	1.54E2	37.83
17.	0.5	0.88	0.97	(0.62; 0.74)	0.07	1.21	4.09	(22.44; 23.94)
	1.0	0.42	12.1	0.81	0.07	0.69	6.72	25.32
18.	0.5	0.78	0.28	(2.88E2; 3.18E2)	0.04	2.38E3	4.10E3	(34.93; 34.95)
	1.0	0.47	4.76E2	3.24E2	0.03	2.55E3	4.39E3	34.96
19.	0.5	0.78	0.98	(0.50; 0.50)	0.12	0.00	1.88	(12.30; 12.62)
	1.0	0.47	0.60	0.51	0.11	0.00	1.93	12.67

Table V. Simulated and approximate values of average costs.

total number of preventive replacements, the total number of corrective replacements, and g. For ET_0 and g a 95% confidence interval is presented, based on the simulation of 3000 cycles. In some cases (marked with an asterisk) no simulation results are available, since the simulation took too much time (more than 24 hours CPU time on a VAX computer).

We also note that the values for ET_0 in table III correspond to the same system parameters $(M, K; \lambda, \alpha)$ as for model 16, but then for varying (r,R).
To give an indication of the possible deviations of ET_0 and g, we present in table VI the number of configurations that yield values of the absolute value of the relative deviation in certain intervals. It should be noticed however, that the simulation results themselves may deviate from the real value, so that the results should be interpreted with caution.

	$\lvert d\rvert$				
(a) ET_0	0-0.2	0.2-0.4	0.4-0.6	0.6-0.8	0.8-1.0
(b) g	0-0.02	0.02-0.04	0.04-0.06	0.06-0.08	0.08-0.10
(a)	52	9	0	3	0
(b)	46	10	6	1	1

Table VI. Values of $\lvert d\rvert$ for 64 different models.

The following conclusions can be drawn from our numerical investigations. The deviation in g is in general much smaller than the deviation in ET_0 and is in practically all cases under 5%. The deviation in ET_0 is in most cases positive, and less than 50%. Finally, it took roughly about 1 minute to obtain the approximation results on a personal computer.

5. Conclusions

The model presented in this paper can be of use to support the maintenance and replacement decisions for systems that are composed of a "large" number (say ≥ 4) of identical components. In particular this model focusses on the compromise between individual component replacements and complete system replacements. As such as it might be used to balance technological improvements of a new system against the investment costs of such a system. The model contains as major decision variable K, the number of doubtful components which triggers a system replacement. Two classes of policies are considered: an A-policy replaces the whole system when a single component fails (or reaches its preventive maintencance age) while the number of doubtful components is greater than or equal to K. A B-policy prescribes a system replacement at the first epoch at which an individual component fails **after** the first moment at which the number of doubtful components has reached level K.

Besides the control parameter K also the parameters r and R which are used in the lifetime model to indicate the boundary of the doubtful age and the preventive maintenance age respectively can be used as control variables.

Under a given choice of those parameters an approximative formula for the average costs per unit time as well as the expected time until system replacement are presented.

Numerical investigations show that this approximation gives fairly good results and certainly can be used to support the decision how to choose the relevant control variables. In particular we note that the approximations improve by increasing number of components.

The validation of the approximations is performed by simulation. As a byproduct this validation reveals that simulation itself is of little use to support the decision process. It took very long simulation runs to obtain confidence intervals of acceptable width for the average costs and the expected time between system replacements.

Appendix A. Approximation of EN(T) by M(ET)

Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables with distribution function F, mean μ and second moment μ_2. The coefficient of variation is defined by $c_\nu^2 := \frac{\mu_2 - \mu^2}{\mu^2}$. Let $\{N(t), t \geq 0\}$ be the associated renewal process, $M(t) = EN(t)$ the renewal function and T a non-negative random variable, not necessarily independent of $\{N(t), t \geq 0\}$ with distribution function F and mean ν.

In this appendix we address the question to what extent EN(T), the expected number of renewals in the stochastic interval [0,T], can be approximated by $M(\nu)$.

In fact ROSS (1987) applies this approximation in reversed direction. He approximates $M(\nu)$ by the sequence $EN(T_k)$, $k \geq 1$ where T_k denotes a random variable, independent of $\{N(t), t \geq 0\}$ with Erlang $(k,\frac{k}{\nu})$-distribution. Ross provides a recursive scheme for the computation of $EN(T_k)$, starting for k = 1 with

$$EN(T_1) = \frac{E(e^{-\lambda X_1})}{1 - E(e^{-\lambda X_1})}$$

and shows that $EN(T_k)$ converges to $M(\nu)$ as $k \to \infty$ under some mild conditions on M(t). Moreover, it is shown that $EN(T_k)$, $k \geq 1$ constitutes an increasing sequence of lower bounds for $M(\nu)$ when the interrenewal distribution function F is DFR.

<u>PROPOSITION A1</u>. If T is independent of $\{N(t), t \geq 0\}$ then

$$|EN(T) - M(\nu)| \leq c_v^2 + 1$$

<u>PROOF</u>. The proof is an immediate consequence of the following well-known inequalities:

$$M(t) \geq \frac{t}{\mu} - 1 \tag{A1}$$

and

$$M(t) \leq \frac{t}{\mu} + c_v^2 \qquad \text{(Lorden's inequality)} \tag{A2}$$

For (A1) we refer to BARLOW and PROSCHAN (1981) (pp. 171) and for (A2) to CARLSSON and NERMAN (1986). □

PROPOSITION A2. If F is DFR and T independent of $\{N(t),\ t \geq 0\}$ then

$$EN(T) \geq M(\nu) \tag{A3}$$

PROOF. Since F is DFR we conclude that M(t) is concave (see BROWN (1980)). Hence (A3) follows from Jensen's inequality. □

REMARK. The reversed inequality holds when M(t) is convex. However, note that convexity of M(t) is not guaranteed by $F \in IFR$.

EXAMPLE A1. Let T_n denote the epoch of n-th renewal in $\{N(t),\ t \geq 0\}$. Then

$$EN(T_n) = n$$

which implies

$$\lim_{n\to\infty} \{M(ET_n) - EN(T_n)\}$$

$$= \lim_{n\to\infty} \{M(n\mu) - n\} = \frac{1}{2}(c_v^2 - 1).$$

EXAMPLE A2. Let T be exponentially distributed with parameter ν^{-1} and assume that T is independent of $\{N(t),\ t \geq 0\}$. Then

$$EN(T) - M(\nu) = \nu^{-1}\tilde{M}(\nu^{-1}) - M(\nu) \tag{A4}$$

where

$$\tilde{M}(s) := \int_0^\infty e^{-st} M(t)\,dt.$$

Now assume that F is a non-positive mixture of exponentials, i.e.

$$1 - F(t) = p_1 e^{-\mu_1 t} + p_2 e^{-\mu_2 t}, \qquad t \geq 0 \tag{A5}$$

with

$$\mu_1, \mu_2 > 0;\ p_1 > 0,\ p_2 < 0 \text{ and } p_1 + p_2 = 1.$$

Then

$$M(t) = \frac{t}{\mu} + \frac{1}{2}(c_v^2 - 1)(1 - \exp\{-(p_1\mu_2 + p_2\mu_1)t\}), \qquad t \geq 0$$

with

$$\mu = \frac{p_1}{\mu_1} + \frac{p_2}{\mu_2}.$$ (see TIJMS (1986), pp. 74)

From (A4) it follows that

$$EN(T) - M(\nu) = \frac{1}{2}(c_v^2 - 1)(e^{-\nu c} - \frac{1}{1 + \nu c}) \tag{A6}$$

where

$$c := p_1\mu_2 + p_2\mu_1.$$

Equation (A6) yields the following proposition.

PROPOSITION A3. Let T be exponentially distributed with parameter ν^{-1} and suppose that T is independent of $\{N(t),\ t \geq 0\}$. If the interrenewal distribution is of the form (A5), then

$$0 \leq EN(T) - M(\nu) \leq -0.1\ (c_v^2 - 1) \leq 0.05$$

PROOF. The proof is an immediate consequence of (A6) and the following inequalities

$$-0.2 \le e^{-x} - (1+x)^{-1} \le 0 \quad \text{for all } x \ge 0$$

and

$$\frac{1}{2} < c_v^2 \le 1.$$

The last inequality is based on the fact that $p_2 < 0$ (see TIJMS (1986), pp. 400). $\square$

Appendix B. Time between preventive and corrective component replacements

In section 3 it was argued that the time between two consecutive preventive (corrective) replacements of an individual component on $[0,T_0)$ can be considered as the entrance time into the absorbing state 2 of a continuous time Markov chain on $\{0,1,2\}$. The transition diagrams are given in figures 3 (for preventive replacements) and figure 4 (for corrective replacements).

In this appendix we derive the probability distribution of these entrance times.

Let us first consider the time between two preventive replacements.

The infinitesimal matrix Q of the corresponding Markov chain is given by

$$Q = \begin{bmatrix} -\nu_0 p_{01} & \nu_0 p_{01} & 0 \\ \nu_1 p_{13} & -\nu_1 & \nu_1 p_{12} \\ 0 & 0 & 0 \end{bmatrix}$$

and the initial distribution of the Markov chain is $(1,0,0)$.

From Neuts (1981) (pp. 45) we conclude that the distribution of the entrance time into the absorbing state 2 is given by

$$1 - F_p(t) = (1,0)\exp(Tt)e \tag{B1}$$

where $e^T = (1,1)$ and

$$T = \begin{bmatrix} -\nu_0 p_{01} & \nu_0 p_{01} \\ \nu_1 p_{13} & -\nu_1 \end{bmatrix}$$

The matrix T is diagonalizable and has eigenvalues

$$\lambda_{1,2} = -\frac{1}{2}(\nu_0 p_{01} + \nu_1) \pm \frac{1}{2}\sqrt{(\nu_0 p_{01} - \nu_1)^2 + 4\nu_0\nu_1 p_{01} p_{13}}$$

with $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_1 \neq \lambda_2$.

A matrix of eigenvectors is given by

$$D = \begin{bmatrix} \nu_0 p_{01} & \nu_0 p_{01} \\ \nu_0 p_{01} + \lambda_1 & \nu_0 p_{01} + \lambda_2 \end{bmatrix}$$

Hence, by noting that

$$\exp(Tt) = D \exp(\Lambda t) D^{-1}$$

with Λ the diagonal matrix with the eigenvalues of T along the diagonal we conclude from (B1)

$$1 - F_p(t) = p_1 e^{-\mu_1 t} + p_2 e^{-\mu_2 t}, \quad t \geq 0 \tag{B2}$$

with

$$p_1 := \frac{\lambda_2}{\lambda_2 - \lambda_1} > 0,\ p_2 := \frac{-\lambda_1}{\lambda_2 - \lambda_1} < 0,\ p_1 + p_2 = 1,\ \mu_1 = -\lambda_1 \text{ and}$$

$$\mu_2 := -\lambda_2.$$

In a similar way we find for the distribution of the time between two consecutive corrective replacements

$$1 - F_c(t) = p_1 e^{-\mu_1 t} + p_2 e^{-\mu_2 t}, \quad t \geq 0 \tag{B3}$$

with

$$p_1 := \frac{\nu_0 p_{03} + \lambda_2}{\lambda_2 - \lambda_1},\ p_2 := \frac{-\nu_0 p_{03} + \lambda_1}{\lambda_2 - \lambda_1},\ \mu_1 := -\lambda_1 \text{ and } \mu_2 := -\lambda_2$$

where λ_1 and λ_2 denote the eigenvalues of the matrix

$$T = \begin{bmatrix} -\nu_0 & \nu_0 p_{01} \\ \nu_1 p_{12} & -\nu_1 \end{bmatrix}$$

and are given by

$$\lambda_{1,2} = -\frac{1}{2}(\nu_0 + \nu_1) \pm \frac{1}{2}\sqrt{(\nu_0 - \nu_1)^2 + 4\nu_0\nu_1 p_{01}p_{12}}.$$

By straightforward calculus it can be shown that the IFR-property

$$\nu_1 p_{13} > \nu_0 p_{03}$$

guarantees that either $p_1 < 0$ or $p_2 < 0$.

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